Real Superalgebras

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Abstract

We will study the mathematics behind different $\mathbb{Z}_2$-graded objects, labeled with the "super" prefix, and those objects endowed with Real and $*$ structures that generalize the notion of complex conjugation and adjoint respectively. We will formulate and compile many useful definitions and then prove properties of these super objects. We will start with a review of some useful linear algebra definitions, then move to super-vector spaces, then to Real Structures, $*$ structures, and superalgebras, and finally conclude with Real super-$*$-modules.

1. Introduction

Supersymmetry is a physical theory that posits a symmetry between bosons and fermions. The mathematics of supersymmetry uses $\mathbb{Z}_2$-grading of objects to account for these two species of particles. Super versions of algebraic objects such as vector spaces, algebras, and modules then follow from this $\mathbb{Z}_2$-grading. Furthermore, the rules for manipulating these super-objects involve subtle signs related to fermionic statistics in physics.

We start with a recollection of standard definitions from linear algebra. These definitions will be generalized to superlinear algebra in the subsequent sections. Throughout, vector spaces will be taken to be vector spaces over the complex numbers $\mathbb{C}$.

Definition 1.1. Let's recall a *direct sum* and *tensor product* of vector spaces. We say a vector space V is a direct sum of vector spaces U, W if U and W are subspaces of V and:

1. $U \cap W = 0$
2. $U + W = V$

We denote a direct sum by writing $V = U \oplus W$
We can define a tenor product in terms of bases. Let $\{v_i\}$ be a basis for V and $\{w_j\}$ be one for W. Then, $V \otimes W$ is the vector space spanned by $v_i \otimes w_j$ such that for all $v, v' \in V$ $w, w' \in W$ $c, c' \in \mathbb{C}$ we have:

1. $(cv + c'v') \otimes w = c(v \otimes w) + c'(v' \otimes w)$
2. $v \otimes (cw + c'w') = c(v \otimes w) + c'(v \otimes w')$

Also, note the symmetry isomorphism:

$$\begin{aligned} \sigma : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w \otimes v. \end{aligned}$$

Definition 1.2. An *algebra* over $\mathbb{C}$ is a vector space A over with a notion of multiplication of two vectors:

$$\begin{aligned} A \times A &\to A \\ (a_1, a_2) &\mapsto a_1 \cdot a_2. \end{aligned}$$

Which has a ring structure compatible with the scalar multiplication by the field.

Definition 1.3. If A and B are algebras over $\mathbb{C}$ and $F : A \to B$ a function, we call F an *algebra homomorphism* if for all $c \in \mathbb{C}$ and $a_1, a_2 \in A$ we have:

1. $f(ca_1) = cf(a_1)$
2. $f(a_1 + a_2) = f(a_1) + f(a_2)$
3. $f(a_1 a_2) = f(a_1)f(a_2)$

Definition 1.4. A hermitian pairing on a complex vector space V is a linear map $h \colon \bar{V} \otimes V \to \mathbb{C}$ such that

$$h(v, w) = \overline{h(w, v)}$$

A hermitian pairing is nondegenerate when $\langle v, w \rangle = 0$ for all w implies that $v = 0$. A hermitian pairing is positive when $\langle v, v \rangle > 0$ for all $v \neq 0$.

2. Supervector Spaces, Real Structures, Inner Products

Definition 2.1. A *supervector space* V over a field k is a vector space over the field k with a $\mathbb{Z}_2$ grading, i.e., V can be written as a direct sum

$$V = V^0 \oplus V^1$$

where V^0 is called the even subspace and V^1 is called the odd subspace.

Notation 2.2. For any field k, we denote the supervector space $k^{p|q}$ as

$$k^{p|q} = k^p \oplus k^q \tag{1}$$

where k^p is the even subspace and k^q is the odd subspace. In particular, when $k = \mathbb{C}$ we have the supervector space $\mathbb{C}^{n|m}$.

Definition 2.3. The direct sum of two supervector spaces V and W is $V \oplus W$ is constructed with the following grading over k:

$$\begin{gathered}(V \oplus W)^0 := V^0 \oplus W^0 \\ (V \oplus W)^1 := V^1 \oplus W^1\end{gathered}$$

Definition 2.4. The tensor product of two supervector spaces V and W is $V \otimes W$ is constructed with the following grading over k:

$$\begin{gathered}(V \otimes W)^0 := V^0 \otimes W^0 \oplus V^1 \otimes W^1 \\ (V \otimes W)^1 := V^1 \otimes W^0 \oplus V^0 \otimes W^1\end{gathered}$$

The notion of degree can be defined for homogeneous vectors v, w, that is, vectors with components only in either V^0 or V^1. If v is a homogeneous vector in V^0, then $deg(v) = 0$, and if w is a homogeneous vector in V^1, then $deg(w) = 1$. Degree is not defined for non-homogeneous vectors. To shorten notation, we define $|v| := deg(v)$.
Note: Given a supervector space (and later a superalgebra), V, unless stated otherwise, when talking about elements in V, we are referring to the homogeneous vectors in V.
Under tensor product, the degree is additive, so we have:

$$deg(v \otimes w) = deg(v) + deg(w).$$

We have the following isomorphism of tensor products in the super world:

$$V \otimes W \xrightarrow{\sim} W \otimes V, \qquad v \otimes w \mapsto (-1)^{|V||w|} w \otimes v.$$

The underlying structure of the direct sum and tensor product of supervector spaces is the same as our usual direct sum and tensor product between vector spaces; a grading is simply built into the construction.

Definition 2.5. Given a supervector space V, the complex vector space $\overline{V}$ is the same as the underlying real vector space V except with scalar multiplication precomposed with complex conjugation.

Definition 2.6. Given a linear map $\varphi : V \to W$, we define $\overline{\varphi} : \overline{V} \to \overline{W}$ as follows.

$$\overline{\varphi}(\overline{v}) = \overline{\varphi(v)}$$

Note that there is a canonical isomorphism between $\overline{V \oplus W} \cong \overline{V} \oplus \overline{W}$ and $\overline{V \otimes W} \cong \overline{V} \otimes \overline{W}$. We will make use of this in section 4.

Definition 2.7. A Real structure is defined on a (non-super) complex vector space V with a $\mathbb{C}$ - antilinear map $r : V \to \overline{V}$ such that $\overline{r} \circ r = id_V$. [1]

Example 2.8. A basic example of a Real structure is complex conjugation,

$$\mathbb{C}^{n|m} \to \overline{\mathbb{C}}^{n|m} \qquad v \mapsto \overline{v}.$$

Remark 2.9. V is a complex vector space with Real structure r. Suppose $a, b \in V$ with a linear map $T : a \to b \; T \in End(V)$. Then $r \circ T = \bar{T} \circ r$.

$$\begin{tikzcd}
a \arrow[r, "T"] \arrow[d, "r"'] & b \arrow[d, "r"] \\
\bar{a} \arrow[r, "\bar{T}"'] & \bar{b}
\end{tikzcd}$$

Definition 2.10. Given a Real vector space (V, r) a hermitian pairing $h\colon \overline{V} \otimes V \to \mathbb{C}$ is *Real* if the following diagram commutes

$$\begin{tikzcd}
\bar{V} \otimes V \arrow[r, "\langle -,- \rangle"] \arrow[d, "\bar{r} \otimes r"] & \mathbb{C} \arrow[d, "\overline{(\;)}"] \\
V \otimes \bar{V} \arrow[r, "\overline{\langle -,- \rangle}"] & \mathbb{C}
\end{tikzcd}$$

Definition 2.11. Given a super vector space V, a hermitian pairing is a linear map $h\colon \bar{V} \otimes V \to \mathbb{C}$ such that

$$h(v, w) = (-1)^{|v||w|}\overline{h(w, v)}$$

Definition 2.12. A hermitian pairing on a super vector space V is positive if for $v \neq 0$,

$$\begin{cases} \langle v, v \rangle > 0, & v \text{ even} \\ -i\langle v, v \rangle > 0, & v \text{ odd} \end{cases}$$

Definition 2.13. Let T be an linear map on a super inner product space V with inner product $\langle -, - \rangle$. Then:
The *ordinary adjoint* of T, denoted $T^\dagger$, has the property that

$$\langle T^\dagger v, w \rangle = \langle v, Tw \rangle \quad \forall\, v, w \in V.$$

The *super-adjoint* of T, denoted T^*, has the property that

$$\langle T^* v, w \rangle = (-1)^{|T||v|}\langle v, Tw \rangle \quad \forall\, v, w \in V.$$

Proposition 2.14. *Suppose we have a super-Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ with super-hermitian form h. Then we can define an ordinary Hilbert space structure $\langle -, - \rangle : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ by taking $\mathcal{H}^0 \perp \mathcal{H}^1$ and*

$$\langle v, w \rangle = \begin{cases} h(v, w), & v, w \in \mathcal{H}^0 \\ -ih(v, w), & v, w \in \mathcal{H}^1 \\ 0, & \textit{else} \end{cases}$$

Proof. To show that $\langle -, - \rangle$ is an ordinary Hilbert space we need to establish the following three properties.

1. Conjugate symmetry: $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all v, w in $\mathcal{H}$
   $v, w \in \mathcal{H}^0 \Rightarrow \langle v, w\rangle = h(v, w) = (-1)^{0 \cdot 0}\overline{h(w, v)} = \overline{\langle w, v\rangle}$
   $v, w \in \mathcal{H}^1 \Rightarrow \langle v, w\rangle = -ih(v, w) = -i(-1)^{1 \cdot 1}\overline{h(w, v)} = i\overline{h(w, v)} = \overline{-ih(w, v)} = \overline{\langle w, v\rangle}$

2. Antilinearity in first argument: $\langle av_1 + bv_2, w\rangle = \overline{a}\langle v_1, w\rangle + \overline{b}\langle v_2, w\rangle$ for all $v_1, v_2, w \in \mathcal{H}$ and $a, b \in \mathbb{C}$
   $v_1, v_2, w \in \mathcal{H}^0 \Rightarrow \langle av_1 + bv_2, w\rangle = h(av_1 + bv_2, w) = \overline{a}h(v_1, w) + \overline{b}h(v_2, w) = \overline{a}\langle v_1, w\rangle + \overline{b}\langle v_2, w\rangle$
   $v_1, v_2, w \in \mathcal{H}^1 \Rightarrow \langle av_1 + bv_2, w\rangle = -ih(av_1 + bv_2, w) = \overline{a}(-i)h(v_1, w) + \overline{b}(-i)h(v_2, w) = \overline{a}\langle v_1, w\rangle + \overline{b}\langle v_2, w\rangle$

3. Positive definiteness: For all $v \in \mathcal{H}$, $v \neq 0$, $\langle v, v\rangle > 0$
   $v \in \mathcal{H}^0 \Rightarrow \langle v, v\rangle = h(v, v) > 0$
   $v \in \mathcal{H}^1 \Rightarrow \langle v, v\rangle = -ih(v, v) > 0$ by definition.

□

Proposition 2.15. *If T is an operator on $\mathcal{H}$, then the super-adjoint T^* and ordinary adjoint $T^\dagger$ (the latter with respect to $\langle -, -\rangle$) are related by*

$$T^* = \begin{cases} T^\dagger, & |T| = 0 \\ iT^\dagger, & |T| = 1 \end{cases}$$

Proof. We wish to show that T^* satisfies the expected property of adjoints.
We first observe that the second argument of $\langle -, -\rangle$ is linear: $\langle v, aw_0 + bw_1\rangle = \overline{\langle aw_0 + bw_1, v_0\rangle} = \overline{\overline{a}\langle w_0, v_0\rangle + \overline{b}\langle w_1, v_0\rangle} = a\overline{\langle w_0, v_0\rangle} + b\overline{\langle w_1, v_0\rangle} = a\langle v_0, w_0\rangle + b\langle v_0, w_1\rangle$. We then use the properties of antilinearity and linearity for the first and second arguments respectively to combine and separate terms in our proof.

1. T even: We want to show that $\langle T^*v, w\rangle = \langle v, Tw\rangle$ and thus $T^* = T^\dagger$
   We have $\langle T^*(v_0 + v_1), w_0 + w_1\rangle$
   $= \langle T^*v_0 + T^*v_1, w_0 + w_1\rangle$
   $= \langle T^*v_0, w_0 + w_1\rangle + \langle T^*v_1, w_0 + w_1\rangle$ (antilinearity of first arg.)
   $= \langle T^*v_0, w_0\rangle + \langle T^*v_0, w_1\rangle + \langle T^*v_1, w_0\rangle + \langle T^*v_1, w_1\rangle$ (linearity of second arg.)
   $= \langle T^*v_0, w_0\rangle + \langle T^*v_1, w_1\rangle$ (definition of $\langle -, -\rangle$)
   $= h(T^*v_0, w_0) - ih(T^*v_1, w_1)$ (definition of $\langle -, -\rangle$)
   $= (-1)^{0 \cdot 0}h(v_0, Tw_0) + (-1)^{0 \cdot 1}(-i)h(v_1, Tw_1)$ (definition of T^*)
   $= \langle v_0, Tw_0\rangle + \langle v_1, Tw_1\rangle$
   $= \langle v_0 + v_1, T(w_0 + w_1)\rangle$

2. T odd: We want to show that $\langle T^*v, w\rangle = -i\langle v, Tw\rangle$ and thus $T^* = iT^\dagger$
   We have $\langle T^*(v_0 + v_1), w_0 + w_1\rangle$
   $= \langle T^*v_0 + T^*v_1, w_0 + w_1\rangle$
   $= \langle T^*v_0, w_0 + w_1\rangle + \langle T^*v_1, w_0 + w_1\rangle$
   $= \langle T^*v_0, w_0\rangle + \langle T^*v_0, w_1\rangle + \langle T^*v_1, w_0\rangle + \langle T^*v_1, w_1\rangle$
   $= \langle T^*v_0, w_1\rangle + \langle T^*v_1, w_0\rangle$
   $= -ih(T^*v_0, w_1) + h(T^*v_1, w_0)$

$$
\begin{aligned}
&= (-1)^{1\cdot 0}(-i)h(v_0, Tw_1) + (-1)^{1\cdot 1}h(v_1, Tw_0)\\
&= -i\langle v_0, Tw_1\rangle - i\langle v_1, Tw_0\rangle\\
&= -i\langle v_0 + v_1, T(w_0 + w_1)\rangle
\end{aligned}
$$

□

Definition 2.16. Consider two supervector spaces $V = V^0 \oplus V^1$ and $W = W^0 \oplus W^1$ and a map $f : V \to W$. We say f is *grading preserving* if

$$f : \begin{cases} V^0 \to W^0 \\ V^1 \to W^1. \end{cases}$$

Furthermore, we say f is *grading reversing* if

$$f : \begin{cases} V^0 \to W^1 \\ V^1 \to W^0. \end{cases}$$

To further illustrate this notation, for $v \in V$, one can think of a grading preserving map as

$$\left(\begin{array}{c|c} v & 0 \\ \hline 0 & v \end{array}\right),$$

and a grading reserving map as

$$\left(\begin{array}{c|c} 0 & v \\ \hline v & 0 \end{array}\right).$$

It is worth noting that through this paper, we often called grading preserving maps even linear transformations and grading reversing maps odd linear transformations.

As convention for a supervector space V over a field k we take the degree of k to be 0. This leads to the following definition.

Definition 2.17. Let V be a supervector space over the field k, the *dual space*, $V^\vee$ in the category of supervector contains the morphisms $V \to k^{1|0}$ [2].

Remark 2.18. The dual space $V^\vee$ for a supervector space has the following $\mathbb{Z}_2$ grading.

$$
\begin{aligned}
(V^\vee)^0 &:= (V^0)^\vee\\
(V^\vee)^1 &:= (V^1)^\vee
\end{aligned}
$$

Definition 2.19. For two supervector spaces V, W, we define the *internal hom* as:

$$\underline{\mathrm{Hom}}(V, W) \cong V^\vee \otimes W.$$

Notice that $\underline{\mathrm{Hom}}(V, W)$ has a natural $\mathbb{Z}_2$ grading where the grading preserving maps are even and the grading reversing maps are odd. If $W = V$ we $\mathrm{End}(V)$, i.e., $\mathrm{End}(V) := \underline{\mathrm{Hom}}(V, V)$.

One should take note that thus far we have only defined the internal hom. The external hom is given by the following definition.

Definition 2.20. For two supervector spaces V, W, the *external hom*, $\mathrm{Hom}(V, W)$ is also a space consisting of morphisms, $V \to W$, but it consists of only even (grading preserving) transformations. I.e., [2]

$$\mathrm{Hom}(V, W) := \underline{\mathrm{Hom}}(V, W)^0.$$

3. Superalgebras, Real Super $*$-structures

Definition 3.1. A *superalgebra* A is a supervector space over a field K equipped with the following morphism of supervector spaces.

$$\begin{aligned} A \otimes A &\xrightarrow{\mu} A \\ a \otimes b &\mapsto ab \end{aligned}$$

With the following properties:

1. Associativity: The following diagram commutes,

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}_A \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

2. Identity: There exists $1 \in A$ such that for all $a \in A$ we have $1a = a1 = a$.

Note since the multiplication map μ is grading preserving, $|ab| = |a| + |b|$. [2]

Definition 3.2. For a superalgebra A, two elements $a, b \in A$ *super-commute* if $ab = (-1)^{|a||b|}ba$. A is called *supercommutative* if every $a, b \in A$ super-commute.

Example 3.3. If V is a supervector space, then $\mathrm{End}(V)$ is a superalgebra with multiplication given by the composition of linear maps.

Definition 3.4. The *opposite superalgebra* of a superalgebra A, denoted A^{op} is a superalgebra that is identical to A except its product. If we denote the product of two elements in A^{op} as $a_1 \cdot^{op} a_2$, then we compute the product as follows.

$$a_1 \cdot^{opp} a_2 = (-1)^{|a_1||a_2|} a_2 a_1.$$

Definition 3.5. If $\varphi : A \to B$ is an algebra homomorphism, φ^{op} can be defined as follows in terms of $a \in A$ and $a^{op} \in A^{op}$. Since A and A^{op} are the same as sets, a and a^{op} are equivalent as objects.

$$\begin{aligned} \varphi^{op} &: A^{op} \to B^{op} \\ \varphi^{op}(a^{op}) &\mapsto \varphi(a)^{op} \end{aligned}$$

Definition 3.6. If A and B are superalgebras, then a *superalgebra homomorphism*, $\varphi : A \to B$ is a homomorphism of algebras that preserves parity, i.e., for all vectors $a, b \in A$ and for any scalar $k \in A$, the following conditions are met.

1. $\varphi(ka) = k\varphi(a)$

2. $\varphi(a+b) = \varphi(a) + \varphi(b)$
3. $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$
4. $|a| = |\varphi(a)|$

If φ is also a bijection, then it is a *superalgebra isomorphism.* If a superalgebra isomorphism exists between two superalgebras A, B, we say that they are isomorphic. This is denoted by $A \cong B$.

Lemma 3.7. *If A, B are superalgebras, then $A^{op} \oplus B^{op} \cong (A \oplus B)^{op}$.*

Proof. We will prove this by showing that

$$\begin{gathered} f : A^{op} \oplus B^{op} \to (A \oplus B)^{op} \\ (a, b) \mapsto (a, b) \end{gathered}$$

is a superalgebra isomorphism. Note that f is trivially well-defined. Next, It is clear that as ungraded algebras, $A^{op} \oplus B^{op}$ and $(A \oplus B)^{op}$ are isomorphic; hence we have an algebra isomorphism. Hence, if we show the preservation of parity, we have a superalgebra isomorphism. First consider $(a_1, b_1), (a_2, b_2) \in A^{op} \oplus B^{op}$. Then take

$$f((a_1, b_1) \cdot (a_2, b_2)) = f((a_1 \cdot_{op} a_2, b_1 \cdot_{op} b_2)) = (-1)^{|a_1||a_2||b_1||b_2|} f((a_2 a_1, b_2 b_1)).$$

Now consider,

$$f((a_1, b_1)) \cdot_{op} f((a_2, b_2)) = (-1)^{|a_1||a_2||b_1||b_2|} f((a_2, b_2)(a_1, b_1)) = (-1)^{|a_1||a_2||b_1||b_2|} f((a_2 a_1, b_2 b_1)).$$

Hence f preserves parity and thus $A^{op} \oplus B^{op}$ is isomorphic to $(A \oplus B)^{op}$. □

Lemma 3.8. *If A, B are superalgebras, then there exists a map,*

$$\begin{gathered} h : A^{op} \otimes B^{op} \to (A \otimes B)^{op} \\ (a, b) \mapsto (a, b) \end{gathered}$$

such that h is a superalgebra isomorphism.

Proof. One can see that h is trivially an algebra isomorphism. To show that h is a superalgebra isomorphism, we must prove that h preserves parity. I.e., for $(a_1 \otimes b_1), (a_2 \otimes b_2) \in A^{op} \otimes B^{op}$, the following equality holds: $h((a_1 \otimes b_1) \cdot (a_2 \otimes b_2)) = h(a_1 \otimes b_2) \cdot_{op} h(a_2 \otimes b_2)$. We will first consider the left side of desired equality.

$$\begin{aligned} h((a_1 \otimes b_1) \cdot (a_2 \otimes b_2)) &= h((-1)^{|a_2||b_1|}(a_1 \cdot_{op} a_2) \otimes (b_1 \cdot_{op} b_2)) \\ = (-1)^{|a_2||b_2|}(-1)^{|a_1||a_2|}(-1)^{|b_2||b_2|}(a_2 a_1 \otimes b_2 b_1) &= (-1)^{|a_1||a_2|+|a_2||b_1|+|b_1||b_2|}(a_2 a_1 \otimes b_2 b_1). \end{aligned}$$

Next, we turn our attention to the other side of the desired equality.

$$\begin{aligned} & h(a_1 \otimes b_2) \cdot_{op} h(a_2 \otimes b_2) = (a_1 \otimes b_1) \cdot_{op} (a_2 \otimes b_2) = (-1)^{|a_1 \otimes b_1||a_2 \otimes b_2|}(a_2 \otimes b_2) \cdot (a_1 \otimes b_1) \\ & = (-1)^{(|a_2|+|b_2|)(|a_1|+|b_1|)}(a_2 a_1 \otimes b_2 b_2) = (-1)^{|a_2||a_1|+|a_2||b_1|+|b_2||a_1|+|b_2||b_1|+|a_1||b_2|}(a_1 \otimes b_1) \\ & = (-1)^{|a_1||a_2|+|a_2||b_1|+|b_1||b_2|}(a_2 a_1 \otimes b_2 b_1). \end{aligned}$$

Hence, parity is preserved, and thus h is a superalgebra isomorphism. □

Definition 3.9. Given a vector space with quadratic form q, a *Clifford algbrea* $Cl(V, q)$ is an algebra satisfying the following universal property where A is an algebra with unity.

$$\begin{tikzcd}
V \arrow[r, hook, "\pi"] \arrow[d, "\varphi"'] & \mathrm{C}\ell(V, q) \arrow[dl, dashed, "\psi"] \\
A &
\end{tikzcd}$$

Where ψ exists unique if $\varphi(v) \cdot_A \varphi(v) = q(v) \cdot 1_A$.

Example 3.10. $\mathrm{C}\ell(V, q)$ is naturally a superalgebra where the image of $V \subset Cl(V, q)$ under φ is odd.

Example 3.11. If $\mathrm{C}\ell(V, q)$ is a Clifford algebra and $q = 0$, then $\mathrm{C}\ell(V, q) \cong \bigwedge V$. Hence the exterior algebra is another example of a superalgebra.

Definition 3.12. We define the Clifford algebra $\mathrm{C}\ell_{m,n} := \mathrm{C}\ell(\mathbb{R}^{m|n}, q)$. Furthermore, we define $\mathbb{C}\ell_{m,n} := \mathrm{C}\ell(\mathbb{C}^{m|n}, q)$.

Definition 3.13. A *Real structure* R on a superalgebra A is a $\mathbb{C}$-antilinear superalgebra homomorphism, $R : A \to \bar{A}$ such that $\bar{R} \circ R = \mathrm{id}_A = R \circ \bar{R}$.

Definition 3.14. A Real superalgebra is a tuple (A, R_A) where A is a superalgebra and R_A is a Real structure.

Definition 3.15. A Real superalgebra homomorphism is a superalgebra homomorphism between two superalgebras with Real structures such that the Real relations are preserved.

Example 3.16. We can define a Real structure R, on the superalgebra $\mathbb{C}\ell_{m,n}$, as the following.

$$\begin{aligned} R : \mathbb{C}\ell_{m,n} = \mathrm{C}\ell_{m,n} \otimes \mathbb{C} &\to \mathrm{C}\ell_{m,n} \otimes \mathbb{C} \\ (v, a) &\mapsto (v, \overline{a}) \end{aligned}$$

Where $\overline{a}$ denotes complex conjugation.

Proposition 3.17. *Given a Real supervector space V, $\mathrm{End}(V)$ naturally admits the following real structure.*

$$\begin{aligned} R_{\mathrm{End}(V)} : \mathrm{End}(V) &\to \overline{\mathrm{End}(V)} \\ T &\mapsto R_V \circ T \circ \overline{R_V} \end{aligned}$$

where R_V is the Real structure on V. Note that canonically $\overline{\mathrm{End}(V)} \cong \mathrm{End}(\overline{V})$.

Proof. We first observe that since R_V is a Real structure on V, $R_{\mathrm{End}(V)}$ squares to the identity map.

$$\left(R_{\mathrm{End}(V)} \circ \overline{R_{\mathrm{End}(V)}}\right)(T) = R_V \circ \overline{R_V} \circ T \circ \overline{R_V} \circ R_V = T$$

Moreover, since R_V is a Real structure, it is $\mathbb{C}$-antilinear. Hence $R_{\mathrm{End}(V)}$ is as well. $\square$

Theorem 3.18. *If (A, R_A) and (B, R_B) are Real superalgebras, then $(A \oplus B, R_{A\oplus B})$ is a Real superalgebra.*

Proof. For $R_{A\oplus B}$ to be a Real structure on $A \oplus B$, we need to show that it squares to $\mathrm{id}_{A\oplus B}$. We will prove this via diagram chase. First, we will show that the following diagram commutes.

$$\begin{tikzcd}
A \oplus B \arrow[r, "R_{A\oplus B}"] \arrow[d, "R_A \oplus R_B"'] & \overline{A \oplus B} \\
\overline{A} \oplus \overline{B} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Note it is clear to see that if (A, R_A) is a Real superalgebra, then $(\overline{A}, \overline{R_A})$ is a Real superalgebra. We are given the Real structures $R_A : A \to \overline{A}$ and $R_B : B \to \overline{B}$ and define

$$\begin{aligned} R_{A\oplus B} : A \oplus B &\to \overline{A \oplus B} \\ (a, b) &\mapsto \overline{(a, b)} \end{aligned}$$

to be a superalgebra homomorphism. Next, define

$$\begin{aligned} \psi : \overline{A} \oplus \overline{B} &\to \overline{A \oplus B} \\ (\overline{a}, \overline{b}) &\mapsto \overline{(a, b)}. \end{aligned}$$

It is clear that ψ is an algebra homomorphism, and since complex conjugation preserves parity, it is a superalgebra homomorphism. Consider $(a, b) \in A \oplus B$ and take

$$\psi(R_A \oplus R_B((a, b))) = \psi((\overline{a}, \overline{b})) = \overline{(a, b)} = R_{A\oplus B}(a, b).$$

Hence $R_{A\oplus B}$ commutes through $\overline{A} \oplus \overline{B}$. Next we show that

$$\begin{tikzcd}
 & \overline{A \oplus B} \arrow[dd, "\overline{R_{A\oplus B}}"'] \\
\overline{A} \oplus \overline{B} \arrow[ur, "\psi"] \arrow[dr, "\overline{R_A} \oplus \overline{R_B}"'] & \\
 & A \oplus B
\end{tikzcd}$$

commutes. Take $(\overline{a}, \overline{b}) \in \overline{A} \oplus \overline{B}$. Then

$$\overline{R_{A\oplus B}}(\psi((\overline{a}, \overline{b}))) = \overline{R_{A\oplus B}}(\overline{(a, b)}) = (a, b) = \overline{R_A} \oplus \overline{R_B}(\overline{a}, \overline{b}).$$

Hence $\overline{R_A} \oplus \overline{R_B}$ commutes through $\overline{A \oplus B}$. Now since ψ is a superalgebra homomorphism, it follows that

$$\begin{aligned} \overline{\psi} : \overline{\overline{A} \oplus \overline{B}} &\to A \oplus B \\ \overline{(\overline{a}, \overline{b})} &\mapsto (a, b) \end{aligned}$$

is also a superalgebra homomorphism. Lastly, we will show that the following diagram commutes.

$$\begin{tikzcd}
\overline{A\oplus B} \arrow[dd, "\overline{R_{A\oplus B}}"'] \arrow[dr, "\overline{R_A\oplus R_B}"] & \\
& \overline{\overline{A}\oplus\overline{B}} \arrow[dl, "\overline{\psi}"] \\
A\oplus B &
\end{tikzcd}$$

Take $\overline{(a,b)}\in\overline{A\oplus B}$ and consider

$$\overline{\psi}(\overline{R_A\oplus R_B}(\overline{(a,b)})) = \overline{\psi}\overline{(\overline{a},\overline{b})} = (a,b) = \overline{R_{A\oplus B}}(\overline{(a,b)})$$

hence $\overline{R_{A\oplus B}}$ commutes through $\overline{\overline{A}\oplus\overline{B}}$. Thus the following diagram commutes and $(R_{A\oplus B})^2 = \mathrm{id}_{R_{A\oplus B}}$.

$$\begin{tikzcd}
A\oplus B \arrow[r, "R_{A\oplus B}"] \arrow[d, "R_A\oplus R_B"'] & \overline{A\oplus B} \arrow[dd, "\overline{R_{A\oplus B}}"'] \arrow[dr, "\overline{R_A\oplus R_B}"] & \\
\overline{A}\oplus\overline{B} \arrow[ur, "\psi"] \arrow[dr, "\overline{R_A}\oplus\overline{R_B}"'] & & \overline{\overline{A}\oplus\overline{B}} \arrow[dl, "\overline{\psi}"] \\
& A\oplus B &
\end{tikzcd}$$

□

Theorem 3.19. *If (A,R_A) and (B,R_B) are Real superalgebras, then $(A\otimes B, R_{A\otimes B})$ is a Real superalgebra with the following multiplication structure.*

$$(a\otimes b)\cdot(a'\otimes b') = (-1)^{|b||a'|}(aa'\otimes bb')$$

Proof. The proof of this proposition is the same as the proof of proposition 3.8 but with $A\otimes B$ instead of $A\oplus B$. □

Definition 3.20. A $*$-structure on a superalgebra A consists of a map $*: A\to\bar{A}^{opp}$ such that $*^2 = \mathrm{id}_A$.

Definition 3.21. A *super-$*$-algebra* is the pair $(A,*_A)$ where A is a superalgebra and $*_A$ is a $*$-strucutre on A.

Remark 3.22. For any super-$*$-algebra $(A,*_A)$ and for $a\in A$, it common to write a^{*_A} instead of $*_A(a)$. We will use both conventions throughout this paper.

Example 3.23. $\mathbb{C}\ell_{m|n}$ has a natural $*$-structure. First note that $\mathbb{C}\ell_{m|n} = C\ell_{m|n}\otimes\mathbb{C}$. Then define the $*$ as

$$*(v) = \begin{cases} iv, & v\in\mathbb{R}^m \\ -iv & v\in\mathbb{R}^n. \end{cases}$$

Definition 3.24. A $*$-homomorphism is a superalgebra homomorphism between two superalgebras with $*$ structures such that the $*$ relations are preserved.

Theorem 3.25. *If $(A, *_A)$ and $(B, *_B)$ are super-$*$-algebras, then $(A \oplus B, *_{A\oplus B})$ is a super-$*$-algebra.*

Proof. For $*_{A\oplus B}$ to be a $*$-structure on $A \oplus B$, we must show that $(*_{A\oplus B})^2 = \mathrm{id}_{A\oplus B}$. We do this via diagram chase. Notice that if $(A, *_A)$ is a super-$*$-algebra then $(\overline{A}, \overline{*_A}^{op})$ is a super-$*$-algebra. Define a superalgebra homomorphism f as

$$\begin{aligned} f : \overline{A}^{op} \oplus \overline{B}^{op} &\to (\overline{A \oplus B})^{op} \\ (\overline{a}, \overline{b}) &\mapsto \overline{(a, b)}. \end{aligned}$$

Since conjugation does not affect parity and by lemma 3.6 f is a superalgebra homomorphism. Next, we will show the following diagram of commutes.

$$\begin{tikzcd}
A \oplus B \arrow[r, "*_{A\oplus B}"] \arrow[d, "*_A \oplus *_B"'] & (\overline{A \oplus B})^{op} \\
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[ur, "f"] &
\end{tikzcd}$$

Consider $(a, b) \in A \oplus B$. Then

$$f(*_B \oplus *_B((a, b))) = f((\overline{a}, \overline{b})) = \overline{(a, b)} = *_{A\oplus B}((a, b)).$$

Hence $*_{A\oplus B}$ commutes through $\overline{A}^{op} \oplus \overline{B}^{op}$. Next we will show that the following diagram commutes.

$$\begin{tikzcd}
 & (\overline{A \oplus B})^{op} \arrow[dd, "\overline{*_{A\oplus B}}"'] \\
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[ur, "f"] \arrow[dr, "\overline{*}_A^{op} \oplus \overline{*}_b^{op}"'] & \\
 & A \oplus B
\end{tikzcd}$$

Note that it since $*_{A\oplus B}$ is assumed to be a superlagebra homomorphism, then

$$\begin{aligned} \overline{*_{A\oplus B}} : (\overline{A \oplus B})^{op} &\to A \oplus B \\ \overline{(a, b)} &\mapsto (a, b) \end{aligned}$$

is also a superalgebra homomorphism. Take $(\overline{a}, \overline{b}) \in \overline{A}^{op} \oplus \overline{B}^{op}$. Then

$$\overline{*_{A\oplus B}}(f((\overline{a}, \overline{b}))) = \overline{*_{A\oplus B}}(\overline{(a, b)}) = (a, b) = \overline{*_A}^{op} \oplus \overline{*_B}^{op}((\overline{a}, \overline{b})).$$

Hence the diagram commutes. Notice that follows that if $f : \overline{A}^{op} \to \overline{B}^{op}$ is a superalgebra homomorphism, then

$$\begin{aligned} \overline{f}^{op} : (\overline{\overline{A} \oplus \overline{B}})^{op} &\to A \oplus B \\ \overline{(\overline{a}, \overline{b})} &\mapsto (a, b) \end{aligned}$$

is a superalgebra homomorphism. Furthermore, since $*_{A\oplus B}$ is assumed to be a superalgebra homomorphism then

$$\begin{aligned} (\overline{*_A \oplus *_B})^{op} : (\overline{A \oplus B})^{op} &\to (\overline{\overline{A} \oplus \overline{B}})^{op} \\ \overline{(a, b)} &\mapsto \overline{(\overline{a}, \overline{b})} \end{aligned}$$

is also a superalgebra homomorphism. Lastly, we show the following diagram commutes.

$$\begin{tikzcd}
(\overline{A \oplus B})^{op} \arrow[dr, "(\overline{*_A \oplus *_B})^{op}"] \arrow[dd, "\overline{*_{A\oplus B}}"'] & \\
& (\overline{\overline{A} \oplus \overline{B}})^{op} \arrow[dl, "\overline{f}^{op}"] \\
A \oplus B &
\end{tikzcd}$$

Take $\overline{(a, b)} \in (\overline{A \oplus B})^{op}$. Then

$$\overline{f}^{op}(\overline{(*_A \oplus *_B)}^{op}(\overline{(a, b)})) = \overline{f}^{op}(\overline{(\overline{a}, \overline{b})}) = (a, b) = \overline{*_{A\oplus B}}(\overline{a, b}).$$

Hence $\overline{*_{A\oplus B}}$ factors through $(\overline{\overline{A} \oplus \overline{B}})^{op}$. Thus

$$\begin{tikzcd}
A \oplus B \arrow[r, "*_{A\oplus B}"] \arrow[dd, "*_A \oplus *_B"'] & (\overline{A \oplus B})^{op} \arrow[dr, "(\overline{*_A \oplus *_B})^{op}"] \arrow[ddd, "\overline{*_{A\oplus B}}"'] & \\
& & (\overline{\overline{A} \oplus \overline{B}})^{op} \arrow[ddl, "\overline{f}^{op}"] \\
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[uur, "f"] \arrow[dr, "\overline{*_A}^{op} \oplus \overline{*_B}^{op}"'] & & \\
& A \oplus B &
\end{tikzcd}$$

commutes which implies $(*_{A\oplus B})^2 = \mathrm{id}_{A\oplus B}$ proving the theorem. ☐

Theorem 3.26. *If* $(A, *_A)$ *and* $(B, *_B)$ *are super-$*$-algebras, then* $(A \otimes B, *_{A\times B})$ *is a super-$*$-algebra with the following multiplication structure.*

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'||b|+|a'||b|+|b||b'|}$$

Proof. We know that $A \otimes B$ is a superalgebra, so we must show that $*_{A\otimes B}$ is a $*$-structure on $A \otimes B$. In other words, we want to prove that $(*_{A\otimes B})^2 = \mathrm{id}_{A\otimes B}$. As in theorem 3.13, we will make use of the fact that if $(A, *_A)$ is a super-$*$-algebra, then $(\overline{A}, \overline{*_A}^{op})$ is also a super-$*$-algebra.
Claim: If $\overline{A}^{op}$ and $\overline{B}^{op}$ are superalgebras, then

$$h : \overline{A}^{op} \otimes \overline{B}^{op} \to (\overline{A \otimes B})^{op}$$
$$(\overline{a} \otimes \overline{b}) \mapsto \overline{a \otimes b}$$

is a superalgebra homomorphism. As ungraded algebras, it is clear that h is a homomorphism. Then, to prove the claim, all that is needed it is to show that parity is preserved under h. Then, by lemma 3.7 and by the fact that conjugation does not affect parity, h preserves parity and is thus a homomorphism. Next, we will show that the following diagram commutes.

$$\begin{tikzcd}
A \otimes B \arrow[r, "*_{A\otimes B}"] \arrow[d, "*_A \otimes *_B"'] & \overline{(A \otimes B)}^{op} \\
\overline{A}^{op} \otimes \overline{B}^{op} \arrow[ur, "h"] &
\end{tikzcd}$$

Let $(a \otimes b) \in A \otimes B$. Then, if we compose h with $*_A \otimes *_B$, we get the following.

$$h(*_A \otimes *_B((a \otimes b))) = h(\overline{a} \otimes \overline{b}) = \overline{a \otimes b} = *_{A\otimes B}((a \otimes b)).$$

Hence $*_{A\otimes B}$ factors through $\overline{A}^{op} \otimes \overline{B}^{op}$. Now, we will show the diagram below commutes.

$$\begin{tikzcd}
& \overline{(A \otimes B)}^{op} \arrow[d, "\overline{*_{A\otimes B}}"'] \\
\overline{A}^{op} \otimes \overline{B}^{op} \arrow[ur, "h"] \arrow[r, "\overline{*_B}^{op} \otimes \overline{*_B}^{op}"'] & A \otimes B
\end{tikzcd}$$

Let $(\overline{a} \otimes \overline{b}) \in \overline{A}^{op} \otimes \overline{B}^{op}$. Then consider the following composition of functions.

$$\overline{*_{A\otimes B}}(h((\overline{a} \otimes \overline{b}))) = \overline{*_{A\otimes B}}(\overline{(a \otimes b)}) = a \otimes b = \overline{*_A}^{op} \otimes \overline{*_B}^{op}((\overline{a} \otimes \overline{b})).$$

Hence, the diagram commutes. Lastly, we must show that:

$$\begin{tikzcd}
\overline{(A \otimes B)}^{op} \arrow[dd, "\overline{*_{A\otimes B}}"'] \arrow[dr, "\overline{(*_A \otimes *_B)}^{op}"] & \\
& \overline{(\overline{A} \otimes \overline{B})}^{op} \arrow[dl, "\overline{h}^{op}"] \\
A \otimes B &
\end{tikzcd}$$

commutes. Let $\overline{(a \otimes b)} \in \overline{(A \otimes B)}^{op}$. Then consider the following composition.

$$\overline{h}^{op}(\overline{(*_A \otimes *_B)}^{op}(\overline{(a \otimes b)})) = \overline{h}^{op}(\overline{(\overline{a} \otimes \overline{b})}) = a \otimes b = \overline{*_{A\otimes B}}(\overline{(a \otimes b)}).$$

Hence $\overline{*_{A\otimes B}}$ commutes through $\overline{(\overline{A} \otimes \overline{B})}^{op}$. Combining the diagrams we get that:

$$\begin{tikzcd}
A \otimes B \arrow[r, "*_{A\otimes B}"] \arrow[dd, "*_A \otimes *_B"'] & \overline{(A \otimes B)}^{op} \arrow[dr, "\overline{(*_A \otimes *_B)}^{op}"] \arrow[ddd, "\overline{*_{A\otimes B}}"'] & \\
 & & \overline{(\overline{A} \otimes \overline{B})}^{op} \arrow[ddl, "\overline{h}^{op}"] \\
\overline{A}^{op} \otimes \overline{B}^{op} \arrow[uur, "h"] \arrow[dr, "\overline{*_A}^{op} \otimes \overline{*_B}^{op}"'] & & \\
 & A \otimes B &
\end{tikzcd}$$

commutes, showing that $(*_{A\otimes B})^2 = \mathrm{id}_{A\otimes B}$ proving the theorem. □

Example 3.27. Given a supervector space V, $\mathrm{End}(V)$ would have a $*$-structure such that

$$* : \mathrm{End}(V) \to \overline{\mathrm{End}(V)}^{op}$$
$$(T_1T_2)^* = (-1)^{|T_1||T_2|}T_2^*T_1^*$$

In the case where V is an inner product space, one such structure is the super-adjoint as defined in definition 2.14.

Thus far, we have discussed both Real and $*$-structures on a superalgebra. It is natural to wonder how one could define a superalgebra that has both a Real and a $*$ structure. Giving us the following definition.

Definition 3.28. We define a *Real super-$*$-algebra* as a tuple $(A, R, *)$ where A is a superalgebra, R is a Real structure on A, and $*$ is a $*$-structure on A such that the following diagram commutes up to a sign.

$$\begin{tikzcd}
A \arrow[r, "R"] \arrow[d, "*"'] \arrow[dr, phantom, "\substack{-1 \\ \circlearrowleft}"] & \overline{A} \arrow[d, "\overline{*}"] \\
\overline{A}^{op} \arrow[r, "\overline{R}^{op}"'] & A^{op}
\end{tikzcd}$$

In other words

$$\overline{R}^{op}(a^*) = (-1)^{|a|}R(a)^{\overline{*}}. \tag{2}$$

Definition 3.29. Let A, B be Real super-$*$-algebras. Then the map $f : A \to B$ is a *super-$*$-homomorphism* if the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "*_A"'] & B \arrow[d, "*_B"] \\ \overline{A}^{op} \arrow[r, "\overline{f}^{op}"'] & \overline{B}^{op} \end{tikzcd}$$

In other words, for every $a \in A$ we have $f(a^{*_A}) = f(a)^{*_B}$.

Proposition 3.30. *Let* $V = \mathbb{C}^{n|m}$*, then* $(\mathrm{End}(V), R_{\mathrm{End}(V)}, *_{\mathrm{End}(V)})$ *forms a Real super-$*$-algebra where the Real and $*$ structures are defined as follows. Letting the Real structure on V be complex conjugation on $\mathbb{C}^n$ and multiplication by i composed with complex conjugation on $\mathbb{C}^m$, we get the following Real structure on* $\mathrm{End}(V)$*.*

$T \to \overline{T}$, $T \in \mathrm{End}(V)^0$
$T \to i\overline{T}$, $T \in \mathrm{End}(V)^1$

We will use the $*$-structure on $\mathrm{End}(V)$ *as defined for arbitrary Real supervector spaces previously in example* 3.24*. Note that since our vector space is $\mathbb{C}^{n|m}$, the the ordinary adjoint in* $\mathrm{End}(V)$ *is complex conjugate composed with transpose. Also note that these two operations commute with one another.*

Proof. Hence, we wish to show our two structures satisfies the relation from definition 3.25. Take $T_0 \in \mathrm{End}(V)^0$ and $T_1 \in \mathrm{End}(V)^1$ and consider the following. Note that we will use R for $R_{\mathrm{End}(V)}$ and $*$ for $*_{\mathrm{End}(V)}$ for brevity.

$$\overline{R}^{op}(T_0^*) = \overline{R}^{op}(T_0^\dagger) = \overline{\overline{T_0^t}} = T_0^t = \left(\overline{\overline{T_0}}\right)^t = \overline{T_0}^{\overline{*}} = R(T_0)^{\overline{*}}$$

Hence we have $\overline{R}^{op} \circ * = (-1)^{|T_0|}\overline{*} \circ R$ as desired. Now consider

$$\overline{R}^{op}(T_1^*) = \overline{R}^{op}(iT^\dagger) = i^2\overline{T^\dagger} = -T^t$$

and

$$R(T_1)^{\overline{*}} = (i\overline{T_1})^{\overline{*}} = i(i\overline{T_1})^\dagger = i\left(\overline{i\overline{T_1}}\right)^t = i(-iT_1)^t = -i^2T_1^t = T_1^t$$

In this case we also have $\overline{R}^{op} \circ * = (-1)^{|T_1|}\overline{*} \circ R$ as desired.
Thus $(\mathrm{End}(V), R_{\mathrm{End}(V)}, *_{\mathrm{End}(V)})$ forms a Real super-$*$-algebra. □

We have previously shown that you can tensor and sum both Real superalgebras and super-$*$-algebras. In the following propositions, we will show that you can sum and tensor Real super-$*$-algebras.

Proposition 3.31. *If* $(A, R_A, *_A)$ *and* $(B, R_B, *_B)$ *are Real super-*$*$*-algebras, then* $(A \oplus B, R_{A\oplus B}, *_{A\oplus B})$ *is a Real super-*$*$*-algebra.*

Proof. Theorem 3.16 tells us $(A \oplus B, R_{A\oplus B})$ is a Real superalgebra, and theorem 3.15 tells us that $(A \oplus B, *_{A\oplus B})$. Then, to prove the proposition, all we need to show is that equation (2) is satisfied. We will show this via diagram chase. Note that it follows directly from the definition of Real super-$*$-algebra that the following diagram commutes up to sign.

$$\begin{tikzcd}
A \oplus B \arrow[rr, "R_A \oplus R_B"] \arrow[d, "*_A \oplus *_B"'] & & \overline{A} \oplus \overline{B} \arrow[d, "\overline{*_A} \oplus \overline{*_B}"] \\
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[rr, "\overline{R_A}^{op} \oplus \overline{R_B}^{op}"'] & {\begin{matrix} -1 \\ \circlearrowleft \end{matrix}} & A^{op} \oplus B^{op}
\end{tikzcd}$$

Next, define the ψ as follows.

$$\psi : \overline{A} \oplus \overline{B} \to \overline{A \oplus B}$$
$$(\overline{a}, \overline{b}) \mapsto \overline{(a, b)}$$

From theorem 3.10, we know that ψ is a superalgebra homomorphism. It is trivial to see that ψ is also a super-$*$-homomorphism. With this, we will now show the following diagram commutes.

$$\begin{tikzcd}
& \overline{A} \oplus \overline{B} \arrow[d, "\psi"] \\
A \oplus B \arrow[ur, "R_A \oplus R_B"] \arrow[r, "R_{A\oplus B}"] & \overline{A \oplus B}
\end{tikzcd}$$

Let $(a, b) \in A \oplus B$. Then consider $(\psi \circ R_A \oplus R_B)(a, b)$.

$$\psi(R_A \oplus R_B((a, b))) = \psi((\overline{a}, \overline{b})) = \overline{(a, b)} = R_{A\oplus B}(a, b)$$

Hence, the diagram commutes. Next, recall that lemma 3.6 tells us

$$h : A^{op} \oplus B^{op} \to (A \oplus B)^{op}$$
$$(a, b) \mapsto (a, b)$$

is a superalgebra homomorphism. Clearly for all $(a, b) \in A^{op} \oplus B^{op}$ we have $h((a, b)^{*_{A\oplus B}}) = h((a, b))^{*_{A\oplus B}}$. Hence, h is a super-$*$-homomorphism. We will now show the following diagram commutes.

$$\begin{tikzcd}
\overline{A} \oplus \overline{B} \arrow[d, "\psi"] \arrow[dr, "\overline{*_A} \oplus \overline{*_B}"] & \\
\overline{A \oplus B} \arrow[d, "\overline{*_{A\oplus B}}"] & A^{op} \oplus B^{op} \arrow[dl, "h"] \\
(A \oplus B)^{op} &
\end{tikzcd}$$

I.e., we have want to show that for $(\overline{a}, \overline{b}) \in \overline{A} \oplus \overline{B}$, we have

$$(\overline{*_{A\oplus B}} \circ \psi)((\overline{a}, \overline{b})) = (h \circ (\overline{*_B} \oplus \overline{*_B}))((\overline{a}, \overline{b})).$$

To start, let us consider the left-hand side of the desired equality.

$$\overline{*_{A\oplus B}}(\psi((\overline{a}, \overline{b}))) = \overline{*_{A\oplus B}}(\overline{(a, b)}) = (a, b) = h((a, b)) = h(\overline{*_B} \oplus \overline{*_B}((a, b))).$$

Thus, we have our desired equality. So far, we have shown that the following diagram commutes.

$$\begin{tikzcd}
 & \overline{A} \oplus \overline{B} \arrow[d, "\psi"] \arrow[dr, "\overline{*_A} \oplus \overline{*_B}"] & \\
A \oplus B \arrow[ur, "R_A \oplus R_B"] \arrow[r, "R_{A\oplus B}"] & \overline{A \oplus B} \arrow[d, "\overline{*_{A\oplus B}}"] & A^{op} \oplus B^{op} \arrow[dl, "h"] \\
 & (A \oplus B)^{op} &
\end{tikzcd}$$

In particular we have shown that for all $(a, b) \in A \oplus B$ we have

$$(\overline{*_{A\oplus B}} \circ R_{A\oplus B})(a, b) = (h \circ \overline{*_A} \oplus \overline{*_B} \circ R_A \oplus R_B)(a, b).$$

Recall from theorem 3.15 that the following diagram commutes.

$$\begin{tikzcd}
 & A \oplus B \arrow[dl, "*_A \oplus *_B"'] \arrow[d, "*_{A\oplus B}"] \\
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[r, "f"'] & \overline{(A \oplus B)}^{op}
\end{tikzcd}$$

Then we will show that

$$\begin{tikzcd}
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[r, "f"'] \arrow[dr, "\overline{R_A}^{op} \oplus \overline{R_B}^{op}"'] & \overline{(A \oplus B)}^{op} \arrow[r, "\overline{R_{A\oplus B}}^{op}"] & (A \oplus B)^{op} \\
& A^{op} \oplus B^{op} \arrow[ur, "h"'] &
\end{tikzcd}$$

also commutes. Let $(\overline{a}, \overline{b}) \in \overline{A}^{op} \oplus \overline{B}^{op}$. Then we have

$$\overline{R^{op}_{A\oplus B}}(f((\overline{a}, \overline{b}))) = \overline{R^{op}_{A\oplus B}}(\overline{a, b}) = (a, b) = h((a, b)) = h(\overline{R_A}^{op} \oplus \overline{R_B}^{op}((\overline{a}, \overline{b}))).$$

So the diagram commutes, giving us the following commuting diagram.

$$\begin{tikzcd}
& A \oplus B \arrow[dl, "*_A \oplus *_B"'] \arrow[d, "*_{A\oplus B}"] & \\
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[r, "f"'] \arrow[dr, "\overline{R_A}^{op} \oplus \overline{R_B}^{op}"'] & \overline{(A \oplus B)}^{op} \arrow[r, "\overline{R_{A\oplus B}}^{op}"] & (A \oplus B)^{op} \\
& A^{op} \oplus B^{op} \arrow[ur, "h"'] &
\end{tikzcd}$$

In particular, we have shown the following equality for all $(a, b) \in A \oplus B$.

$$(h \circ \overline{R_A}^{op} \oplus \overline{R_B}^{op} \circ *_A \oplus *_B)(a, b) = (\overline{R_{A\oplus B}}^{op} \circ *_{A\oplus B})(a, b)$$

Now we recall that from the definition of Real super-$*$-algebra, we have

$$(\overline{R_A}^{op} \oplus \overline{R_B}^{op} \circ *_A \oplus *_B)(a, b) = (-1)^{|a||b|}(\overline{*_A} \oplus \overline{*_B} \circ R_A \oplus R_B)$$

Then, using the equalities above, we have the following.

$$\begin{aligned}
&(\overline{R_A}^{op} \oplus \overline{R_B}^{op} \circ *_A \oplus *_B)(a, b) = (-1)^{|a||b|}(\overline{*_A} \oplus \overline{*_B} \circ R_A \oplus R_B)(a, b) \\
\iff &(h \circ \overline{R_A}^{op} \oplus \overline{R_B}^{op} \circ *_A \oplus *_B)(a, b) = (-1)^{|a||b|}(h \circ \overline{*_A} \oplus \overline{*_B} \circ R_A \oplus R_B)(a, b) \\
\iff &(\overline{R_{A\oplus B}}^{op} \circ *_{A\oplus B})(a, b) = (-1)^{|a||b|}(\overline{*_{A\oplus B}} \circ R_{A\oplus B})(a, b)
\end{aligned}$$

Prove that we have satisfied equation (2). The proof can be summarized by the following diagram commuting up to sign.

$$\begin{tikzcd}
 & & \overline{A} \oplus \overline{B} \arrow[d, "\psi"] \arrow[dr, "\overline{*_A} \oplus \overline{*_B}"] & \\
 & A \oplus B \arrow[ur, "R_A \oplus R_B"] \arrow[r, "R_{A \oplus B}"] \arrow[dl, "*_A \oplus *_B"'] \arrow[d, "*_{A \oplus B}"] & \overline{A \oplus B} \arrow[d, "\overline{*_{A \oplus B}}"] & A^{op} \oplus B^{op} \arrow[dl, "h"] \\
\overline{A}^{op} \oplus \overline{B}^{op} \arrow[r, "f"'] \arrow[dr, "\overline{R_A}^{op} \oplus \overline{R_B}^{op}"'] & \overline{(A \oplus B)}^{op} \arrow[r, "\overline{R_{A \oplus B}}^{op}"] & (A \oplus B)^{op} & \\
 & A^{op} \oplus B^{op} \arrow[ur, "h"'] & &
\end{tikzcd}$$

□

Proposition 3.32. *If* $(A, R_A, *_A)$ *and* $(B, R_B, *_B)$ *are Real super-$*$-algebras, then* $(A \otimes B, R_{A\otimes B}, *_{A\otimes B})$ *is a Real super-$*$-algebra.*

Proof. Using the same reasoning as in the previous proposition, we get that the following diagram commutes up to a sign.

$$\begin{tikzcd}
 & & \overline{A} \otimes \overline{B} \arrow[d, "\psi"] \arrow[dr, "\overline{*_A} \otimes \overline{*_B}"] & \\
 & A \otimes B \arrow[ur, "R_A \otimes R_B"] \arrow[r, "R_{A \otimes B}"] \arrow[dl, "*_A \otimes *_B"'] \arrow[d, "*_{A \otimes B}"] & \overline{A \otimes B} \arrow[d, "\overline{*_{A \otimes B}}"] & A^{op} \otimes B^{op} \arrow[dl, "h"] \\
\overline{A}^{op} \otimes \overline{B}^{op} \arrow[r, "f"'] \arrow[dr, "\overline{R_A}^{op} \otimes \overline{R_B}^{op}"'] & \overline{(A \otimes B)}^{op} \arrow[r, "\overline{R_{A \otimes B}}^{op}"] & (A \otimes B)^{op} & \\
 & A^{op} \otimes B^{op} \arrow[ur, "h"'] & &
\end{tikzcd}$$

□

4. Real super-$*$-Modules

Definition 4.1. A Real super-$*$-module over a Real super-$*$-algebra A with underlying Real super-$*$-inner space V is a Real $*$-homomorphism $A \to \mathrm{End}(V)$ where $\mathrm{End}(V)$ has the usual super $*$-structure, a Real structure defined via the Real structure on V, and a positive Hermian pairing.

Example 4.2. Recall that $\mathrm{End}(V)$ as a Real super-$*$-algebra is the following tuple $(\mathrm{End}(V), R_{\mathrm{End}(V)}, *_{\mathrm{End}(V)})$. We can define a Real super-$*$-module over V via the identity map.

$$\begin{aligned} \mathrm{id} : \mathrm{End}(V) \times V &\to \mathrm{End}(V) \\ (T, v) &\mapsto T(v) \end{aligned}$$

Proposition 4.3. *We can define a Real super-$*$-module α over $C\ell_4$, the Clifford algebra generated by $\mathbb{H}$. We define α as a Real super-$*$-algebra homomorphism*

$$\alpha : C\ell_4 \to \mathrm{End}(\mathbb{R}^{4|4})$$

where $\alpha(x)$, $x \in \mathbb{H}$ acts on an element in $\mathbb{R}^{4|4}$, say (v, w) as follows,

$$x \cdot (v, w) = (-xw, x^* v)$$

where $*$ is the usual adjoint on the quaternions. That is, for $q = a + bi + cj + dk \in \mathbb{H}$, $q^* := a - bi - cj - dk$. Note that $qq^* = q^*q = a^2 + b^2 + c^2 + d^2$.

Proof. We will show that this definition satisfies the universal property of Clifford Algebras. This will show that α is the unique Real super-$*$-algebra homomorphism between $\mathrm{C}\ell_4$ and $End(\mathbb{R}^{4|4})$. Note that we only need to define α on the generators of $\mathbb{C}\ell_4$ to form the Real super-$*$-algebra homomorphism.

$$x \cdot (x \cdot (v, w)) = x \cdot (-xw, x^* v) = (-xx^* v, -x^* xw)$$

By the definition of the usual adjoint on the quaternions, we have

$$\alpha(x)^2 (v, w) = (-||x||^2 v, -||x||^2 w) = ||x||^2 \circ 1_{\mathrm{End}(\mathbb{R}^{4|4})}$$

as desired. □

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